Lecture 27.

- $(M, \mathcal{D})$ strictly ψcvx CR mfld, $n = \text{CR} \dim M$.
- Fix contact form θ.
- Choose local frame $Z_1, \dots, Z_n$ for $\mathcal{D}$. Levi form:

$$g_{\alpha\bar\beta} = \frac{1}{i} d\theta(Z_\alpha, \overline{Z_\beta}). \qquad \alpha, \beta, \dots \in \{1, \dots, n\}.$$

- Choose $\theta^1, \dots, \theta^n$ s.t. $(\theta, \theta^\alpha)^\perp = \overline{\mathcal{D}}$, $\theta^\alpha(Z_\beta) = \delta_\beta^\alpha$, $\theta^{\bar\alpha} = \overline{\theta^\alpha}$ $\Rightarrow$ coframe $(\theta, \theta^\alpha, \theta^{\bar\alpha})$ determined up to

$$\begin{pmatrix} \hat\theta \\ \hat\theta^\alpha \\ \hat\theta^{\bar\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ t^\alpha & t_\beta{}^\alpha & 0 \\ \overline{t^\alpha} & 0 & \overline{t_\beta{}^\alpha} \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\beta \\ \theta^{\bar\beta} \end{pmatrix}. \qquad (*)$$

A

- Coframe $(\theta, \theta^\alpha, \theta^{\bar\alpha}) \rightsquigarrow$ frame $(T, Z_\alpha, Z_{\bar\alpha})$, where T is determined by choice θ^α.

- A change $(*) \Rightarrow (T, Z_\beta) = (\hat T, \hat Z_\alpha) A$, where A is matrix in $(*)$. $\Rightarrow$

- $g_{\alpha\bar\beta} = \hat g_{\nu\bar\mu} t_\alpha{}^\nu \overline{t_\beta{}^\mu}$

<u>Pseudohermitian structures</u> (Webster, Tanaka)

Recall <u>formal integrability</u> $\Rightarrow$
$[Z_\alpha, Z_\beta]$ and $[Z_{\bar\alpha}, Z_{\bar\beta}]$ sections of $\mathcal{V}$ and $\overline{\mathcal{V}}$ respectively $\Rightarrow$ $d\theta$ cannot have $\theta^\alpha \wedge \theta^\beta$ or $\theta^{\bar\alpha} \wedge \theta^{\bar\beta}$ forms (by Cartan's formula, since $d\theta(Z_\alpha, Z_\beta) = \frac{1}{2}\theta([Z_\alpha, Z_\beta])$)

$$\Rightarrow \quad d\theta = i g_{\alpha\bar\beta}\, \theta^\alpha \wedge \theta^{\bar\beta} + \theta \wedge \varphi$$

some real 1-form.

Making the change $(\hat\theta, \hat\theta^\alpha, \hat\theta^{\bar\alpha}) \to (\theta, \theta^\alpha, \theta^{\bar\alpha})$ given by (*) with $t_\beta{}^\alpha = \delta_\beta{}^\alpha$

$$\Rightarrow \left(d\theta = d\theta, \ g_{\alpha\bar\beta} = g_{\alpha\bar\beta} \right)$$

$$i\hat g_{\alpha\bar\beta}\, t_\gamma{}^\alpha\, \overline{t_\mu{}^\beta}\, \theta^\gamma \wedge \theta^{\bar\mu} + \theta \wedge \left(\hat\varphi + i\hat g_{\alpha\bar\beta}\, t^\alpha \theta^{\bar\beta} - i\hat g_{\alpha\bar\beta}\, t^{\bar\beta}\theta^\alpha \right) = i\hat g_{\alpha\bar\beta}\, \theta^\alpha \wedge \theta^{\bar\beta} + \theta \wedge \varphi$$

$$\Rightarrow \hat{\varphi} + i\hat{g}_{\alpha\bar{\beta}} t^{\alpha} \theta^{\bar{\beta}} - i\hat{g}_{\alpha\bar{\beta}} t^{\bar{\beta}} \theta^{\alpha} = \varphi \mod \theta$$

If we expand φ (which is real-valued) in the coframe, we get

$$\varphi = c_\alpha \theta^\alpha + \overline{c_\alpha}\, \theta^{\bar{\alpha}} \mod \theta$$

$\Rightarrow$ We get $\hat{\varphi} = 0 \mod \theta$ if we can solve the linear equation.

$$i\hat{g}_{\alpha\bar{\beta}} t^{\alpha} = \overline{c_\beta},$$

which we can, of course, since $\hat{g}_{\alpha\bar{\beta}}$ is an invertible matrix. Thus, by making a change of coframe $(*)$, we can achieve (calling now $\hat{\theta}^\alpha \Leftrightarrow \theta^\alpha$)

$$d\theta = i g_{\alpha\bar{\beta}}\, \theta^\alpha \wedge \theta^{\bar{\beta}}.$$

Such a coframe $(\theta, \theta^\alpha, \theta^{\bar{\alpha}})$ is called <u>admissible</u> (for the φ-herm structure θ).

If we want to fix a Levi form $g_{\alpha\bar\beta}$, then any two admissible coframes are related by (*) w/ $t^\alpha = 0$ and $(t_\alpha{}^\beta)$ unitary (for $g_{\alpha\bar\beta}$).

An admissible coframe $(\theta, \theta^\alpha, \theta^{\bar\alpha})$ yields the dual frame $(T, Z_\alpha, Z_{\bar\alpha})$ as above.

Changing admissible coframe $\hat\theta^\alpha = t_\beta{}^\alpha \theta^\beta \Rightarrow \hat Z_\beta = t_\beta{}^\alpha \hat Z_\alpha$ but $\hat T = T$.

Def. The vector field T is called the <u>Reeb vector field</u> of the contact form θ. It is uniquely determined by

$$\theta(T) = 1 \ , \ \underbrace{T \lrcorner d\theta}_{\substack{\text{the 1-form } \omega \text{ s.t.} \\ \omega(X) = d\theta(T, X),\ \forall X.}} = 0$$

Next, we let θ^α be some admissible coframe.